Exact Vibration and Buckling Solution of Levy Type Initially Stressed Rectangular Thin Plates

M. Razavi Kermanshahi
Electromechanics R&D Group,
Iranian academic center for education, culture and research, Tehran, Iran
E-mail: mrazavi@jdevs.com

A. A. Lotfi Neyestanak *
Department of Engineering,
Shahr-Rey Branch, Islamic Azad University, Tehran, Iran
E-mail: aklotfi@gmail.com
*Corresponding author

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Abstract: This paper has focused on vibration and buckling of thin rectangular plates when at least two opposite sides are simply supported condition and subjected to axial and biaxial initial stresses. The relevant analysis is accomplished based on an exact solution method, free from any approximate method. First analytic techniques are used to discuss the free vibrations of plate. Then, natural frequencies and critical buckling loads are calculated. It is believed that through the present work, the exact closed-form characteristic equations and their associated eigenfunctions for the considered six cases are obtained for the first time.

Keywords: Buckling, Exact Solution, Free Vibration, In-Plane Buckling Moments, Rectangular Plate, Thin Plate

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Biographical notes: M. Razavi Kermanshahi, is a researcher of Mechanical Engineering at Electro-mechanics R&D Group, Iranian academic center for education, culture and research, Tehran, Iran. His current research interests include vibration analysis of plate, Stress Analysis of oil pipe, Tank and pressure vessel, and Air Pollution Control Equipment. **A. A. Lotfi neyestanak** is an Assistant Professor at the Department of Mechanical Engineering, Shahr-Rey Branch, Islamic Azad University, Tehran, Iran. His current research interests include Air Pollution Control Equipment, WEDM, Conductive polymers and Damage Analysis of Composite.

1 INTRODUCTION

Rectangular plates are commonly used in structural components in many branches of modern technology namely mechanical, aerospace, electronic, marine, optical, nuclear and structural engineering. Thus, the knowledge of their free vibration and buckling load are very important to the structural designers.

Numerous works concerning vibration of such plates have been published so far mostly based on thin plate theory without initial stress. An excellent reference source in this connection may be found in the well-known work of Leissa [1] and his subsequent articles (1977-1987) [2-7] published in vibration Digest from time to time.

His remarkable works on the free vibration of thin rectangular plate [8] also present comprehensive and accurate analytical results for sixth distinct case without in-plane stresses that have exact solution. Liew et al. [9], [10] investigated the free vibration of rectangular plates, respectively, using two dimensional polynomials and one-dimensional Gram-Schmidt polynomials as the admissible functions of the plate in Rayliegh-Ritz method.

The buckling loads [11] of plates which are subjected to edge loads acting in their mid plane are areas of research which have received a great deal of attention in the past century, but they were obtained using an approximate method. Exact solution for Mindlin rectangular plate is another work for plates, also exact solutions for vibration and buckling rectangular plate by in-plane stress is one of comprehensive and accurate analytical results for SS-C-SS-C case [12], [13]. The exact characteristic equation for rectangular thin plates having two opposite sides simply supported can be found in the original work of Leissa [8]. There is not such an equation about thin plates that has been initially stressed in the literature. To fill this apparent void, the present work is carried out to provide the exact characteristic equations for the six cases having two opposite sides simply supported subjected to in-plane loads through the thickness of that, in forms of axial and biaxial. The considered six cases are namely s-c-s-c, s-s-s-c, s-s-s-s, s-c-s-f, s-s-s-f and s-f-s-f boundary conditions.

The current paper acts as the first work dealing with free vibration and buckling the thin rectangular plates for six cases that have initially in-plane stresses (in forms of axial and biaxial) by exact solution. Accurate free vibration frequencies and buckling loads are presented for some important cases for some of initially stressed loading and aspect ratio.

2 GOVERNING EQUATIONS AND THEIR DIMENSIONLESS FORMS

Consider a flat, isotropic, rectangular thin plate of length a, width b, modulus of elasticity E, Poisson's ratio ν, shear modulus $G = E/2(1+\nu)$ and density per unit area ρ, oriented so that its mid-plane surface contains the x_1 and x_2 axis of a Cartesian co-ordinate system (x_1, x_2, x_3).

The displacements along the x_1 and x_2 axes are denoted by u_1 and u_2, respectively while the displacement in the direction perpendicular to the undeformed middle surface is denoted by u_3. In the classic Kirchhoff's plate theory, the displacement components are assumed to be given by:

$$u_1 = -x_3\psi_{3,1} \tag{1a}$$

$$u_2 = -x_3\psi_{3,2} \tag{1b}$$

$$u_3 = \psi_3 \tag{1c}$$

Where ψ_3 is transverse deflection along the x_3 direction. Using the small deflection, the strain components may be expressed as:

$$\varepsilon_1 = -x_3\psi_{3,11} \tag{2a}$$

$$\varepsilon_2 = -x_3\psi_{3,22} \tag{2b}$$

$$\varepsilon_{12} = -x_3\psi_{3,12} \tag{2c}$$

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0 \tag{2d}$$

Based on the strain-displacement relations given in equation (2, a, b, c, d) and assuming a stress distribution in accordance with Hook's low, as well as neglecting the stress strain relations involving ε_{13}, ε_{23} and ε_{33} the resultant bending moments and twisting moment all per unit length in terms of ψ_3 are obtained by integrating the stresses and moment of the stresses through the thickness of the plane. These are given by:

$$M_{11} = D(\psi_{3,11} + V\,\psi_{3,22}) \qquad (3a)$$

$$M_{22} = -D(\psi_{3,11} + \psi_{3,22}) \qquad (3b)$$

$$M_{12} = -D(1-V)\psi_{3,12}) \qquad (3c)$$

Where D is the flexural rigidity. The governing differential equations based on the Navier for plate [14-15] can be:

$$M_{11,1} + M_{12,2} - Q_1 = 0 \qquad (4a)$$

$$M_{12,1} + M_{22,2} - Q_2 = 0 \qquad (4b)$$

$$Q_{1,1} + Q_{2,2} + p = \rho\frac{\partial^2\psi_3}{\partial t^2} \qquad (4c)$$

Where p is the transverse force per unit area due to components of the in-plane loads, in addition p will be expressed as:

$$p = N_1\psi_{3,11} + N_2\psi_{3,22} \qquad (5a)$$

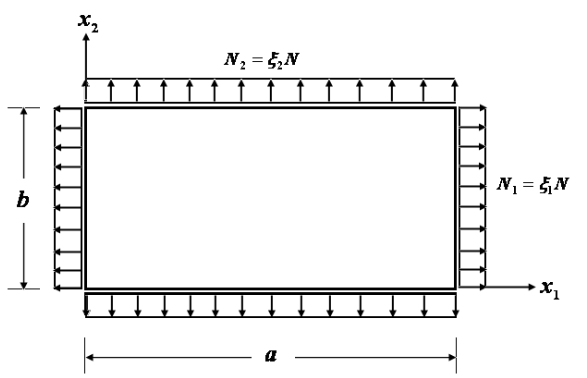

Fig. 1 A rectangular thin plate subjected to in-plane edge loads

The plate is subjected to in-plane edge loads per unit length N_1 in the x_1 direction and N_2 in the x_2 direction, as shown in Fig. 1. The two edges of the plate parallel to the x_2-axis are assumed to be simply supported while the other two edges may have any combinations of clamped, free or simply supported boundary conditions as shown in Fig. 2.

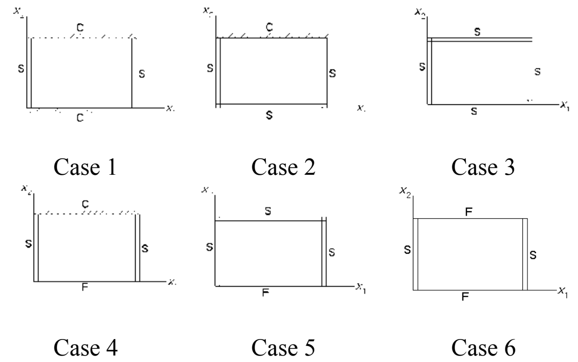

Fig. 2 Boundary conditions of thin plates analyzed

Assuming the free harmonic motion, the governing differential equations for free vibration of the plate under investigation can be presented in terms of ψ_3. In addition, by substituting the stress resultants from expressions (3a)-(3c) into equations (4a)-(4c) will result in:

$$D\left(\psi_{3,11} + \psi_{3,22}\right)_{,1} + Q_1 = 0 \qquad (6a)$$

$$D\left(\psi_{3,11} + \psi_{3,22}\right)_{,2} + Q_2 = 0 \qquad (6b)$$

$$Q_{11} + Q_{22} + N_1\psi_{3,11} + N_1\psi_{3,22} = \rho\omega^2\psi_3 \qquad (6c)$$

Also transverse shearing forces are:

$$V_1 = -D\left(\psi_{3,111} + (2-V)\psi_{3,112}\right) \qquad (7a)$$

$$V_1 = -D\left(\psi_{3,222} + (2-V)\psi_{3,211}\right) \qquad (7b)$$

For generality and convenience, the coordinates are normalized with respect to the plate planar dimensions and the following non-dimensional terms are introduced.

$$X_1 = \frac{x_1}{a},\ X_2 = \frac{x_2}{b},\ \eta = \frac{a}{b}$$

$$\tilde{\psi}_3 = \frac{\bar{\psi}_3}{a},\ (\tilde{V}_1, \tilde{V}_2) = (V_1, V_2)\frac{a}{b} \qquad (8a,b,c,d,e)$$

$$(\bar{M}_{11}, \bar{M}_{22}, \bar{M}_{12}) = \left(M_{11}, M_{22}, M_{12}\right)\frac{a}{D} \qquad (8f)$$

$$\left(\tilde{N}_1,\tilde{N}_2\right)=\left(N_1,N_2\right)\frac{a^2}{D}$$
$$\beta=\omega a^2\sqrt{\frac{\rho}{D}} \quad (8g,h)$$

Where β is the frequency parameter. Substitution of the dimensionless stress resultants (8a)-(8h) into the equations (6a)-(6c) leads to:

$$\left(\tilde{\psi}_{3,11}+n^2\tilde{\psi}_{3,22}\right)_{,1}+Q_1=0 \quad (9a)$$

$$n\left(\tilde{\psi}_{3,11}+n^2\tilde{\psi}_{3,22}\right)_{,2}+Q_2=0 \quad (9b)$$

$$\tilde{Q}_{11}+\tilde{Q}_{2,2}+\tilde{N}_1\tilde{\psi}_{3,11}+N^2\tilde{N}_2\tilde{\psi}_{3,22}=B^2\tilde{\psi}_3 \quad (9c)$$

In order to solve the three coupled partial differential equations (9a)-(9c), it will be more convenient to have a single equation on the transverse deflection ψ_3. This can be obtained by differentiating equations (9a) and (9b) with respect to X_1 and X_2, respectively, then multiplying the latter by η, summing them up and noting equation (9c), we obtain:

$$(\tilde{\psi}_{3,11}+\eta^2\tilde{\psi}_{3,22})_{,11}-\eta^2(\tilde{\psi}_{3,11}+\eta^2\tilde{\psi}_{3,22})_{,22}$$
$$+\tilde{N}_1\tilde{\psi}_{3,11}+\eta^2\tilde{N}_2\tilde{\psi}_{3,22}=\beta^2\tilde{\psi}_3 \quad (10)$$

The solution of the equation (10) can be assumed to be:

$$\psi_3=W_1+W_2=f_2(x_1)g_1(x_2)+f_2(x_1)g_2(x_2) \quad (11)$$

Substituting the above solution into equation (10) yields:

$$\frac{f_{1,1111}}{f_{12}}-\tilde{N}_1\frac{f_{1,11}}{f_1}+n^4\frac{g_{1,2222}}{g_1}-n^2\tilde{N}_2\frac{g_{1,22}}{g_1}$$
$$+2n^2\frac{f_{1,11}}{f_1g_1}g_{1,22}-\beta^2=0 \quad (12a)$$

$$\frac{f_{2,1111}}{f_{22}}-\tilde{N}_1\frac{f_{2,11}}{f_2}+n^4\frac{g_{2,2222}}{g_2}-n^2\tilde{N}_2\frac{g_{2,22}}{g_2}$$
$$+2n^2\frac{f_{2,11}}{f_2g_2}g_{2,22}-\beta^2=0 \quad (12b)$$

Inspection of equations (12a) and (12b) suggest that the functions $f_i(X_1)$ and $g_i(X_2)(i=1,2)$ should be characterized by the equations:

$$f_{i,11}=\pm\mu_i^2f_i \quad , \quad g_{i,22}=\pm\lambda_i^2g_i \quad (13a,b)$$

Where μ_i^2 and λ_i^2 are separation constants to be obtained. By examining the boundary conditions it can be easily shown that a solution to the equations $f_{i,11}=\mu_i^2f_i$ is not suitable for satisfying the boundary conditions when two opposite edges at X_1=0 and X_1=1 are simply supported. Hence, the following solutions to equations (12a) and (12b) can be selected.

$$f_i(x_1)=\tilde{a}_i\sin u_ix_1+\tilde{b}_i\cos u_ix_1 \quad (14)$$

$$g_1(x_1)=\tilde{c}_1\sin\lambda_1x_1+\tilde{d}_1\cos\lambda_1x_1 \quad (15)$$

$$g_2(x_1)=\tilde{c}_2\sin\lambda_2x_1+\tilde{d}_2\cos\lambda_2x_1 \quad (16)$$

As it was discussed in an earlier paper by Hashemi and Arsanjani [12-16], no loss of generality may arise due to selection of any possible set of solutions. The derivation, therefore, may be maintained based on the set of selected solutions as:

$$f_i(x_2)=\tilde{a}_i\sin u_ix_2+\tilde{b}_i\cos u_ix_2 \quad (17)$$

$$g_1(x_2)=\tilde{c}_1\sin\lambda_1x_2+\tilde{d}_1\cos\lambda_1x_2 \quad (18)$$

$$g_2(x_2)=\tilde{c}_2\sin\lambda_2x_2+\tilde{d}_2\cos\lambda_2x_2 \quad (19)$$

Which in turn suggest that:

$$f_{i,11}=-\mu_i^2f_i \quad (20a)$$

$$g_{1,22}=-\lambda_1^2g_1 \quad (20b)$$

$$g_{2,22}=\lambda_2^2g_2 \quad (20c)$$

May be solved by representing the three dimensionless functions, $\tilde{\psi}_3$ in terms of the two dimensionless potentials W_1, and W_2 as:

$$\tilde{\psi}_3=W_1+W_2 \quad (21)$$

It may also be noted that, as $W_1=f_1(X_1)g_1(X_2)$ and $W_2=f_2(X_1)g_2(X_2)$, the relationship between λ_i, μ_i and α_i for the set of selected solutions may be expressed as:

$$\alpha_1^2 = \mu_1^2 + \eta^2 \lambda_1^2 \quad , \quad \alpha_2^2 = \mu_2^2 - \eta^2 \lambda_2^2 \tag{22a,b}$$

Substituting now from equations (20a, b, c) into equations (12a) and (12b) gives:

$$\alpha_1^4 + \tilde{N}_2 \alpha_1^2 + (\tilde{N}_1 - \tilde{N}_2)\mu_1^2 - \beta^2 = 0 \tag{23}$$

$$\alpha_2^4 + \tilde{N}_2 \alpha_2^2 + (\tilde{N}_1 - \tilde{N}_2)\mu_2^2 - \beta^2 = 0 \tag{24}$$

3 TWO OPPOSITE EDGES SIMPLY SUPPORTED

For the sake of definiteness, the dimensionless boundary conditions will be given below for an edge parallel to the X_2-normalized axis (for example, the boundaries $X_1 = 0 \;\; or \;\; X_1 = 1$)

For a simply supported edge:

$$\tilde{M}_{22} = (\nu \tilde{\psi}_{3,11} + \eta^2 \tilde{\psi}_{3,22}) = 0 \, , \tilde{\psi}_3 = 0 \tag{25a,b}$$

For a free edge:

$$\tilde{M}_{22} = (\nu \tilde{\psi}_{3,11} + \eta^2 \tilde{\psi}_{3,22}) = 0 \tag{26a}$$

$$\tilde{V}_2 = -\tilde{N}_2 \eta \tilde{\psi}_{3,2} \tag{26b}$$

For a clamped edge:

$$\tilde{\psi}_2 = \eta \tilde{\psi}_{3,2} = 0 \tag{27a}$$

$$\tilde{\psi}_3 = 0 \tag{27b}$$

Corresponding boundary conditions for the edges $X_2 = 0$ and $X_2 = 1$ are obtained by interchanging subscripts 1 and 2 in equation (25a)-(27b). On the assumption of a simply-supported edge at both $X_1 = 0$ and $X_1 = 1$, equations (17)-(19) may be written as:

$$W_1 = \left[A_1 \sin(\lambda_1 X_2) + A_2 \cos(\lambda_1 X_2)\right] \sin(\mu X_1) \tag{28a}$$

$$W_2 = \left[A_3 \sinh(\lambda_2 X_2) + A_4 \cosh(\lambda_2 X_2)\right] \sin(\mu X_1) \tag{28b}$$

Where $\mu = m\pi; m = 1,2,\ldots$. Introducing equation (28a,b) in equations (21) and substituting the results into the two appropriate boundary conditions along the edges $X_2 = 0$ and $X_2 = 1$ lead to a characteristic determinant of the two order for each m. Expanding the determinant and collecting terms yields a characteristic equation. The characteristic equations for the six cases are listed below. In addition, the six different boundary conditions given in the characteristic equations are shown in Fig. 2.

Case1. S-C-S-C

$$-2\lambda_1\lambda_2 + \lambda_1\lambda_2 \cos\lambda_1 \cosh\lambda_2 + \lambda_1^{\,2} \cos\lambda_1 \sinh\lambda_2 - \lambda_1^{\,2} \sin\lambda_1 \cosh\lambda_2 = 0 \tag{29}$$

Case2. S-C-S-S

$$\lambda_1 \sinh\lambda_2 \cos\lambda_1 - \lambda_2 \sin\lambda_1 \cosh\lambda_2 = 0 \tag{30}$$

Case3. S-S-S-S:

$$\sin\lambda_1 \sinh\lambda_2 = 0 \tag{31}$$

Case4. S-C-S-F

$$\frac{n}{\lambda_2}(A_1 \sin\lambda_1 \sinh\lambda_2 + A_2 \cos\lambda_1 \cosh\lambda_2 + A_3) = 0 \tag{32a}$$

Where

$$A_1 = \mu^2 \nu(\lambda_1^{\,2}(-N_2 - 2\mu^2 + \mu^2\nu - \eta^2\lambda_1^{\,2}) + \lambda_2^{\,2}(N_2 + 2\mu^2 - \mu^2\nu - \eta^2\lambda_2^{\,2})) + \eta^2\lambda_1^{\,2}\lambda_2^{\,2}(2N_2 + 4\mu^2 - 2\mu^2\nu + \eta^2\lambda_1^{\,2} - \eta^2\lambda_2^{\,2} \tag{32b}$$

$$A_2 = \lambda_1\lambda_2(-2N_2\mu^2\nu + 4\mu^4\nu + 2\mu^4\nu^2 - N_2\eta^2\lambda_1^{\,2} - 2\eta^2\mu^2\lambda_1^{\,2} + N_2\eta^2\lambda_2^{\,2} + 2\eta^2\mu^2\lambda_2^{\,2} + 2\eta^4\lambda_1^{\,2}\lambda_2^{\,2}) \tag{32c}$$

$$A_3 = N_2\mu^2\nu\lambda_1\lambda_2 + 2\mu^4\nu\lambda_1\lambda_2 - \mu^4\nu^2\lambda_1\lambda_2 + N_2\eta^2\lambda_1^{\,3}\lambda_2 + 2\eta^2\mu^2\lambda_1^{\,3}\lambda_2 + \eta^4\lambda_1^{\,5}\lambda_2 + N_2\mu^2\nu\lambda_1\lambda_2 + \eta^4\lambda_1\lambda_2^{\,5} - \mu^4\nu^2\lambda_1\lambda_2 - N_2\eta^2\lambda_1\lambda_2^{\,3} - 2\eta^2\mu^2\lambda_1\lambda_2^{\,3} + 2\mu^4\nu\lambda_1\lambda_2 \tag{32d}$$

Case 5. S-S-S-F

$$B_1 \eta Cos\lambda_1 Sinh\lambda_2 + B_2 \eta Cosh\lambda_2 Sin\lambda_1 = 0 \tag{33a}$$

Where

$$B_1 = \mu^2 \nu \lambda_1 (-N_2 + \mu^2 (-2+\nu) - \eta^2 (\lambda_1{}^2 + \lambda_2{}^2)) + \lambda_1{}^2 \lambda_2 \ (N_2 \eta^2 + 2\eta^2 \mu^2 + \eta^4 \lambda_2{}^2) \quad (33b)$$

$$B_2 = \mu^2 \nu \lambda_2 (N_2 + \mu^2 (2-\nu) - \eta^2 (\lambda_1{}^2 + \lambda_2{}^2)) + \lambda_1{}^2 \lambda_2 \ (N_2 \eta^2 + 2\eta^2 \mu^2 - \eta^4 \lambda_2{}^2) \quad (33c)$$

Case 6. S-F-S-F

$$-n(C_1 + C_2 \cos^2 \lambda_1 - \sin^2 \lambda_1 + C_3 \cos \lambda_1 \cosh \lambda_2 + C_4 \sin \lambda_1 \sinh \lambda_2) = 0$$

Where

$$C_1 = \frac{n^3 \lambda_1^3 + 2n\lambda_1 U^2 - \nu n \lambda_1 U^2 + N_2 n \lambda_1}{n^3 \lambda_2^3 + 2n\lambda_2 U^2 - \nu n \lambda_2 U^2 + N_2 n \lambda_2} \quad (34b)$$

$$C_2 = \frac{VU^2 + n^2 \lambda_1^2}{VU^2 + n^2 \lambda_2^2} \quad (34c)$$

$$C_3 = N_2 (U^2 V \lambda_1 + n^2 \lambda_1{}^3) \quad (34d)$$

$$C_4 = 2U^4 V \lambda_1 - U^4 V^2 \lambda_1 + 2n_2 U^2 \lambda_1{}^3 + n^4 \lambda_1{}^5 + C_1 C_2 N_2 U^2 V^2 \lambda_2 - C_1 C_2 N_2 U^4 V^2 \lambda_2 - C_1 C_2 N_2 n^2 \lambda_2{}^3 + 2C_1 C_2 U^4 V \lambda_2{}^3 + 2C_1 C_2 N_2 U^4 V \lambda_2 + C_1 C_2 n^4 \lambda_2{}^5) \quad (34c)$$

4 VIBRATION AND BUCKLING CRITERIA

In order to investigate the exact vibration as well as obtaining the exact critical buckling load parameter of plates for all six combinations of edge conditions as illustrated in Fig. 2, we assume:

$$\tilde{N}_1 = \xi_1 \tilde{N}_{cr} \ , \ \tilde{N}_2 = \xi_2 \tilde{N}_{cr} \quad (35a,b)$$

For given values of ξ_1, ξ_2, η and ν the characteristic equations given for individual cases are functions of β, m and $\tilde{N}_{cr}$. Depending on selection of ξ_1 and ξ_2 which may be either (-1, 0), (0,-1) or (-1,-1), the critical buckling load parameter $\tilde{N}_{cr}$ may be determined by setting $\beta = 0$ in the corresponding characteristic equation of each individual case. Upon testing the different integer values of m (usually $m = 1,2,3$ or $m = 4$) the lowest value of the solved equation should be selected. Having obtained $\tilde{N}_{cr}$, the frequency parameters β_{mn} may be determined by substituting corresponding $\tilde{N}_{cr}$ together with any desired values of ξ_1 and ξ_2 between -1 to 0 into the characteristic equations of each individual case.

5 BUCKLING AND VIBRATION RESULTS

Table 1 presents the non-dimensional buckling load parameter $\tilde{N}_{cr}$ for SCSC, SSSC, SSSS, SCSF, SSSF and SFSF three combinations of $\tilde{N}_1$ and $\tilde{N}_2$ ratio, namely:

$a) \tilde{N}_1 = \tilde{N}, \tilde{N}_2 = 0 \ (\xi_1 = -1, \xi_1 = 0)$

$b) \tilde{N}_1 = 0, \tilde{N}_2 = -\tilde{N} \ (\xi_1 = 0, \xi_1 = -1)$

$c) \tilde{N}_1 = \tilde{N}_2 = -\tilde{N} \ (\xi_1 = \xi_1 = -1)$

are considered. These combinations in-plane loading cover the case of a biaxial in-plane loading in the x_1 and the x_2 directions.

In this part the exact results of critical buckling for a rectangular plate with at least two opposite sides with simply supported condition are presented. To study the effect of the boundary conditions on the buckling of thin plates, the critical buckling parameters listed in table 1 have been arranged. According to the results presented in this table, it is clear that the lowest critical buckling parameters correspond to plates subject to less edge restraints.

A rise in the number of supported edges entails the critical buckling parameters increase. It can be seen that the lowest and highest values of critical buckling parameters correspond to S-F-S-F and S-C-S-C cases respectively. Thus, higher constraints at the edges increase the flexural rigidity of the plate, resulting in a higher critical buckling response rate.

In order to study the effects of aspect ratio, according to the table 1 for a stiffly condition, as the aspect ratio η enhances, the critical buckling parameters also increase. Also by increasing η, for highly constrained plates critical buckling parameters happens in higher value of m. In addition, regarding to table 1, when $(\xi_1, \xi_2) = (-1,0)$, critical buckling parameters have had higher values than the state of $(\xi_1, \xi_2) = (-1,-1)$. The maximum value of buckling parameter happens in the case $(\xi_1, \xi_2) = (0,-1)$.

It means that, when the buckling load is in X1-direction critical buckling parameters are more than other cases and for in-plane loading condition in X2-direction (i.e. the direction that has the simply supported condition) the values of critical buckling parameter is higher than the condition in which all sides are exposed to in-plane loading.

Table 1 Buckling load parameters, $\tilde{N}_{cr} = N_{cr} a^2 / D$ $\left(\tilde{N}_1 = \xi_1 \tilde{N}_{cr}, \tilde{N}_2 = \xi_2 \tilde{N}_{cr}\right)$, for rectangular thin plates

Case	(ξ_1,ξ_2)	η=0.4		η=0.5		η=1		η=2		$\eta=2.5$	
		$\tilde{N}_{cr}$	m	$\tilde{N}_{cr}$	m	$\tilde{N}_{cr}$	m	$\tilde{N}_{cr}$	m	$\tilde{N}_{cr}$	m
S-C-S-C	(-1,0)	14.9196	1	18.9775	1	75.9099	2	275.2280	3	431.6794	4
	(0,-1)	44.6340	1	47.8394	1	66.5526	1	179.5020	1	267.6430	1
	(-1,-1)	12.5665	1	14.6174	1	37.7996	1	150.9920	1	231.8530	2
S-S-S-C	(-1,0)	13.9892	1	16.9094	1	56.6536	1	221.3000	3	334.5290	3
	(0,-1)	41.1222	1	41.8144	1	47.8394	1	102.5100	1	147.2170	1
	(-1,-1)	11.9065	1	13.2373	1	26.2798	1	85.0810	1	130.1790	1
S-S-S-S	(-1,0)	13.2805	1	15.4213	1	39.4784	1	157.9140	2	255.0330	3
	(0,-1)	40.8053	1	39.4784	1	39.4784	1	61.6850	1	83.0040	1
	(-1,-1)	11.4487	1	12.3370	1	19.7392	1	49.3480	1	71.5546	1
S-C-S-F	(-1,0)	10.5183	1	11.0127	1	16.3096	1	52.7423	1	85.4476	2
	(0,-1)	22.8011	1	22.8029	1	23.6093	1	25.9190	1	30.1830	1
	(-1,-1)	9.7607	1	9.8706	1	11.2865	1	18.1064	1	23.5115	1
S-S-S-F	(-1,0)	10.3888	1	10.7474	1	13.8332	1	26.3770	1	35.8145	1
	(0,-1)	22.7995	1	22.7999	1	23.3497	1	20.1630	1	18.1680	1
	(-1,-1)	9.7176	1	9.7954	1	10.4138	1	11.7435	1	12.2142	1
S-F-S-F	(-1,0)	9.6516	1	9.6046	1	9.3989	1	9.1682	1	9.1137	1
	(0,-1)	22.7400	1	22.3118	1	20.1625	1	15.6134	1	14.9785	1
	(-1,-1)	9.4574	1	9.4006	1	9.2004	1	9.0583	1	9.0331	1

Other results may be obtained from table 1 as well. For example, when $(\xi_1,\xi_2) = (-1,0)$, the effect of increasing η on the increment of critical buckling parameter, in the highest edge restrained case is faster than the lower edge restrained (for example in s-c-s-c case $\eta = 0.4$ $\tilde{N}_a = 12.5665$ and $\eta = 2.5$ $\tilde{N}_a = 231.8530$ and in s-c-s-f case $\eta = 0.4$ $\tilde{N}_a = 10.5183$ and $\eta = 2.5$ $\tilde{N}_a = 85.4476$

Moreover in freely boundary conditions, buckling happens in smaller values of m. The table 1 indicates that when $(\xi_1,\xi_2) = (-1,-1)$, effect of η in buckling parameters value's growth is more considerable than when $(\xi_1,\xi_2) = (-1,0)$. In the case $(\xi_1,\xi_2) = (0,-1)$ by increasing η, buckling parameters values possess a slower increasing rate.

Based on the presented results, the lowest frequency parameters correspond to plates subject to less edge restraints. As the number of supported edges rises, the frequency parameters also increase, and it demonstrates that the lowest and highest values of frequency parameters correspond to S-F-S-F and S-C-S-C cases respectively. Thus higher constraints at the edges increase the flexural rigidity of the plate, resulting in a higher frequency response rate.

6 COMPARISION WITH PUBLISHED WORKS

In this section, the results of table (2) are compared with the values of critical buckling for a plate with scsc condition.

Table 2 Comparison of non-dimensional critical buckling loads $\tilde{N}_{cr}/\eta^2$ for s-c-s-c case

Method	0.4	m	0.5	m	0.6	m	1	m
Ref[11]	93.2	1	75.9	1	69.6	1	75.9	2
Ref[13]	93.247	1	75.910	1	69.632	1	45.910	2
present	93.2475	1	75.91	1	69.6322	1	75.9099	2

In tables (3) and (4), frequency values with s-s-s-f and s-f-s-f boundaries are compared for a case without any in-plane load.

Table 3 Comparison study of frequency parameters $\beta = \omega a^2 \sqrt{\rho / L}$ for rectangular thin plate with s-s-s-f boundaries

h=a /b	Meth od	1	2	3	4	5	6	7
	Ref	10.12	13.05	18.83	27.55	39.33	39.61	42.69
0.4	[8]	59	70	90	80	77	18	64
	Prese	10.12	13.05	18.83	27.55	39.33	39.61	42.69
	nt	59	70	90	80	77	18	64
	Ref	10.67	18.29	33.69	40.13	48.40	57.59	64.72
2/3	[8]	12	95	74	07	82	29	81
	Prese	10.67	18.29	33.69	40.13	48.40	57.59	64.72
	nt	12	95	74	07	82	29	81
	Ref	11.68	27.76	41.19	59.06	61.86	90.29	94.48
1	[8]	45	02	87	56	06	31	37
	Ref[9	11.69	27.76	41.19	59.06	61.86	90.29	94.48

	]	25	02	87	56	17	31	17
	PRESE	11.68	27.75	41.19	59.06	61.86	90.29	94.48
	NT	45	63	67	55	06	41	37
1.5	REF	13.71	43.57	47.85	81.47	92.69	124.5	13289
	[8]	11	23	71	89	25	635	74
	PRESE	13.71	43.57	47.85	81.47	92.69	124.5	13289
	NT	11	23	71	89	25	635	74
2.5	REF	18.80	50.54	100.2	110.2	147.6	169.1	203.7
	[8]	09	05	321	259	317	026	304
	PRESE	18.80	50.54	100.2	110.2	147.6	169.1	203.7
	NT	09	05	321	259	317	026	304

Table 4 Comparison study of frequency parameters $\beta = \omega a^2 \sqrt{\rho / D}$ for rectangular thin plate with s-f -s-f boundaries

H=A/B	METHOD	1	2	3	4	5	6	7
0.4	REF	9.76	11.03	15.06	21.70	31.17	39.238	40.503
	[8]	00	68	26	64	71	7	5
	PRESE	9.76	11.03	15.06	21.70	31.17	39.238	40.503
	NT	00	68	26	64	71	7	5
2/3	REF	9.69	12.98	22.95	39.10	40.35	42.684	54.240
	[8]	83	13	35	52	60	7	0
	PRESE	9.69	12.98	22.95	39.10	40.35	42.684	54.240
	NT	83	13	35	52	60	7	0
1	REF	9.63	16.13	36.72	38.94	46.73	70.740	75.283
	[8]	14	48	97	50	81	1	4
	REF[9	9.64	16.14	36.72	38.94	46.73	70.739	75.283
	]	06	17	97	74	95	4	4
	REF[1	9.63	16.13	36.72	38.94	46.73	70.735	75.285
	0]	27	68	48	55	26	5	3
	PRESE	9.63	16.13	36.72	38.94	46.73	70.740	75.283
	NT	14	48	56	50	81	1	4
1.5	REF	9.55	21.61	38.72	54.84	65.79	87.626	103.96
	[8]	82	92	14	43	22	2	65
	PRESE	9.55	21.61	38.72	54.84	65.79	87.626	103.96
	NT	82	92	14	43	22	2	65
2.5	REF	9.48	33.62	38.36	75.20	86.96	130.35	155.32
	[8]	41	28	29	37	84	76	11
	PRESE	9.48	33.62	38.36	75.20	86.96	130.35	155.32
	NT	41	28	29	37	84	76	11

7 CONCLUSION

The present work investigated free vibrations of thin plates based on the Kirchhoff's plate theory. The foregoing work has shown how an exact procedure may be followed to analyze the buckling and free vibration of rectangular plates with initial stresses in forms of one-axial and biaxial. The exact characteristic equations are derived for the six cases having two opposite sides simply supported. The considered cases are s-c-s-c, s-s-s-c, s-s-s-s, s-c-s-f, s-s-s-f and s-f-s-f plates.

The obtained results are all for buckling load and frequency parameter based on exact solution. The results for buckling case clearly show the effect of the boundary condition, aspect ratios and different initial stresses on the plate responses. Also accurate frequency parameters are presented for different aspect ratios and different initial stresses for each case. These frequency parameters may be considered as an exact database for each of considered case and may contribute to people desiring to investigate the accuracy of an approximate method on some of these problems. The effect of boundary condition, variation of aspect ratios and different initial stresses on the frequency values are examined and discussed in detail. In addition, contour plots at any desired frequency parameters are graphically displayed.

Finally, in comparison with previously published works, the validity of the presented results are confirmed. Eigen frequency and critical loads that were given are dimensionless and as a result independent of material property and dimensions of plate. Consequently by considering material property and dimensions of plate, Eigen frequencies and critical loads are usable. The exact results should provide engineers and researchers who work on plate's vibration with a useful reference source. ultimately, its noticeable that the mentioned example is considered as a model for airplane descending at the runway , in this case, the impact caused by loading(impact loads) besides the runway (the sheet placed on the elastic base of Pasternak under single and dual-axis stress) are modelled with different boundary conditions.

8 NOMENCLATURE

ε	Strain
Ss	Simply Support
M	Bending Moments
C	Clamp
β	Frequency Parameter
P	Transverse Force Per Unit Area
E	Modulus Of Elasticity
Y	Poisson's Ratio
G	Shear Modulus
ψ	Transverse Deflection
F	Free
D	Flexural Rigidity
X	Direction.
B	Width Of Rectangular Thin Plate
A	Length Of Rectangular Thin Plate
ρ	Density
U	Displacements
N	Plane Edge Loads Per Unit Length
$\widetilde{N}_{cr}$	Critical Buckling Load Parameter

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