

# Buckling Analysis of Cracked Columns by XFEM

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**Abstract:** This paper focuses on using the recently developed extended finite element model for buckling analysis of edge cracked columns under concentric axial load. The effect of crack depth and its location on the carrying capacity of columns is studied. The effect of different boundary conditions is also investigated. Numerical examples are offered to show the efficiency and effectiveness of the proposed method. The presented results are compared with analytical and experimental works available in the literature. Good agreement with experiments is shown, although the difference with analytical results is considerable for columns with deeper cracks. The reason of this difference is discussed. It is shown that the proposed method is more accurate than the analytical methods which are developed based on rotational spring models.

**Key words:** Buckling, Column, Crack depth, Spring model, XFEM

**References:** Shirazizadeh, M. R., Shahverdi, H. and Imam, A., 'Buckling Analysis of Cracked Columns by XFEM', Int J of Advanced Design and Manufacturing Technology, Vol. 9/No. 1, 2016, pp. 93–102.

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## 1 INTRODUCTION

Stability is one of the main problems in solid mechanics and losing it can cause catastrophic and dangerous collapse in structures. Buckling is one of the most important forms of instabilities. This phenomenon often occurs in slender columns. Buckling of perfect columns under different boundary conditions and loads has been studied by many researchers. A comprehensive study can be found in the work established by Timoshenko and Gere [1]. However, Compression members usually contain various kinds of imperfections like edge cracks and flaws. These imperfections can significantly reduce carrying capacity of columns. Therefore, their effects should be certainly taken into account in the stability analysis of structures.

Many researchers have focused on studying the effect of crack existence on the vibration and stability of structures. For example Vanderveldt and Liebowitz [2] presented analytical and experimental solution for single and double edge cracked columns with concentric and eccentric compressive loads. The experimental and theoretical results were found to agree to within less than 5 percent.

One of the most popular analytical methods used in structural analysis of edge cracked slender members is the local flexibility method. In this method it is assumed that a crack divides a column into two intact parts which are connected together with a massless rotational spring in the location of crack. This method has widely implemented by researchers to study cracked slim structures. For example, Ke et al. [3] studied post buckling response of beams made of functionally graded materials (FGMs) containing an open edge crack. Their method was based on Timoshenko beam theory and the concept of local flexibility. The effect of a crack on the buckling behavior of beam-columns is investigated by Challamel and Xiang [4]. Attar illustrates an analytical approach to investigate natural frequencies and mode shapes of a stepped beam with an arbitrary number of transverse cracks and general form of boundary conditions [5]. Gruel and Kisa [6] used the combination of transfer matrix method and fundamental solution of intact columns for determining the buckling load of cracked columns with rectangular cross sections. Later Gurel [7] expanded this work to study the stability of cracked columns with circular cross section. In these mentioned works, the flexibility of massless rotational spring, that represents an edge crack, was considered as a function of crack depth. The attempts have been focused on developing more accurate local flexibility function of a spring. For example Yazdchi and Gowhari [8] developed new spring flexibility functions based on

two different methods (finite element and J-integral approaches) to study the buckling of prismatic edge cracked columns with circular and rectangular cross sections.

Almost all of the mentioned methods were based on analytical solution of governing equations. Therefore, they can be applied just for special cases with a simple geometry and boundary conditions. In fact, these methods cannot deal with most practical engineering problems which are geometrically complicated. Moreover, it is shown in this paper that, the results of rotational spring models are reasonable just for short cracks. In fact when the length of crack is considerable compared to the thickness of the column, rotational spring methods cannot model the problem with good accuracy.

Classical finite element method is a very powerful tool for modeling complex solid mechanics problems. This method can be used as an alternative solution to avoid the mentioned analytical methods deficiencies. This method has been used by some researchers to study the stability analysis of cracked slim structures. For example, Shen and Pierre [9] studied the effect of crack existence on the natural frequencies of Euler-Bernoulli beams by using Eight-Node isoparametric finite element. Although FEM is the most powerful numerical technique and a wide range of commercial FEM packages are currently used in industries and research centers, improving of its efficiency to model cracks and discontinuities has always been considered as a great challenge. Typically a very refined grid and a use of singular elements around a crack tip are necessary in order to obtain accurate solutions in fracture mechanics problems. If the crack is moving (for example in crack propagation problems) or is located within a complex geometry, an acceptable mesh generating could become extremely time consuming.

The extended FEM (XFEM) is one of the most important refinements in standard FEM to study cracked domains with minimum computational efforts. The philosophy behind this method is the incorporation of special local enrichment functions into a standard FEM approximation. XFEM has been proposed by Belytschko and Black [10] employing the concept of partition of unity. Later moës et al. [11] improved this method to model crack propagation problems without regenerating the mesh around the discontinuity. Recently this method has been employed to solve the wide range of engineering problems. For example, linear buckling of cracked plates has been studied by Baiz et al. [12] using SFEM and XFEM. Bachene et al. [13] used XFEM for analyzing vibration of cracked plates. Natarajan et al.

[14] studied the natural frequency of a FGM rectangular plate by XFEM.

Although the extended finite element method is a developing approach and its implementation in different fields is rapidly increased, the use of this method in buckling analysis of cracked beam structures is not reported in the literature.

In this paper a special extended finite element model, which has been recently introduced by Shirazizadeh and Shahverdi [15], is used for buckling analysis of cracked columns under concentric vertical load. This element is capable to model cracked beam columns with minimum computational cost. The effect of crack depth, crack location and different boundary conditions on the buckling load of columns is investigated. Presented results are compared with analytical and experimental results available in the literature. It is found that the presented results have a very good agreement with the experiment for a wide range of crack depths, though their disagreement is considerable with analytical results when deeper cracks are considered. The reason of this difference is discussed extensively.

**2 A SPECIAL EXTENDED FINITE ELEMENT MODEL FOR MODELING CRACKED COLUMNS**

Shirazizadeh and Shahverdi [15] developed a special XFEM for structural analysis of cracked beam columns with arbitrary cross-section. This element is developed based on combining Timoshenko beam element with only displacement degrees of freedom and partition of unity enrichment. The most important advantage of this model compared to available standard and extended finite element models is its computational efficiency in modeling edge cracked columns. The governing equations of this element are briefly discussed here.

Fig. 1 shows the element and its degrees of freedom with and without crack. As evident, the element has two nodes located in two sides and each node has one degree of freedom in *y* direction (*v*). Also each node has *n* degrees of freedom in *x* direction ( $u_i^j$ ).

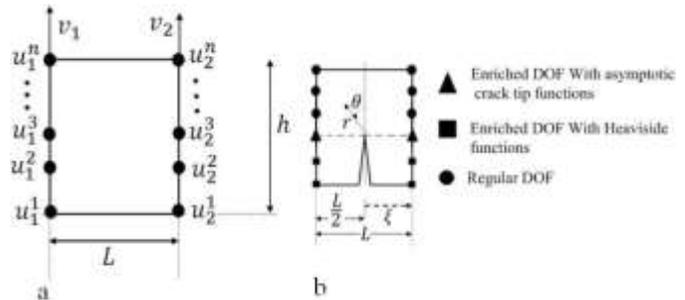


Fig. 1 Timoshenko beam element with only displacement degrees of freedom a) without and b) with edge crack

Therefore the displacement field for intact beam can be written as Eq. (1) [15].

$$\begin{cases} u(x, y) = \sum_{i=1}^2 \sum_{j=1}^n N_i(x) \cdot H_j(y) \cdot u_i^j \\ v(x) = \sum_{i=1}^2 N_i(x) \cdot v_i \end{cases} \quad (1)$$

Where *H(y)* and *N(x)* are the element shape functions. The form of these shape functions are introduced as below:

$$\begin{cases} N_1(x) = 0.5 - \frac{x}{L} \\ N_2(x) = 0.5 + \frac{x}{L} \end{cases} \quad (2)$$

In the above equations *x* and *y* are coordinates located in the middle of the element. It should be noted that, the form of *H(y)* is depend on the selected number of degrees of freedom in *x* direction. These functions can easily be obtained in compliance with the standard finite element shape functions requirements. Eq. (3) shows the form of these functions for some of possible degrees of freedom [15]. The minimum number of required degrees of freedom in the axial direction (*n*) for modeling edge crack depends on the geometry of cracked body. In general, deeper cracks cause more nonlinearity in the distribution of the axial displacement. Therefore, to model deeper cracks, more numbers of *n* should be considered. However, to find an appropriate grid for modeling an edge crack, a mesh study shall be done and a problem should be solved by different numbers of *n*. When the results become independent mesh and they converge to the particular value, the mesh grid would be considered as a suitable one.

$$\left\{ \begin{array}{l} \text{for } n = 2 : \begin{cases} H_1(y) = 0.5 - \frac{y}{h} \\ H_2(y) = 0.5 + \frac{y}{h} \end{cases} \\ \text{for } n = 3 : \begin{cases} H_1(y) = -\frac{y}{h} + \frac{2y^2}{h^2} \\ H_2(y) = 1 - \frac{4y^2}{h^2} \\ H_3(y) = \frac{y}{h} + \frac{2y^2}{h^2} \end{cases} \end{array} \right. \quad (3)$$

For modeling the edge cracks within the element the extended finite element enrichment is used. The extended finite element method is based on adding some special enrichment functions in to the standard finite element approximation for elements which are directly involved with the tip and the side of a crack. Therefore, the displacement field in the element containing an edge crack shown in Fig. 1 is expressed as Eq. (4) [15].

$$\left\{ \begin{array}{l} u(x, y) = \sum_{i=1}^2 \sum_{j=1}^n N_i(x) \cdot H_j(y) \cdot u_i^j + \sum_{i=1}^2 \sum_{j=1}^n N_i(x) \cdot H_j(y) \cdot \tilde{\phi}_{ij}(\xi) \cdot a_i^j + \sum_{i=1}^2 \sum_{j=1}^n \sum_{k=1}^4 N_i(x) \cdot H_j(y) \cdot \tilde{F}_{ijk}(r, \theta) \cdot b_{ik}^j \\ v(x) = \sum_{i=1}^2 N_i(x) \cdot v_i \end{array} \right. \quad (4)$$

In the above equation,  $a_i^j$  are the enriched degrees of freedom associated with the shifted Heaviside function  $\tilde{\phi}_{ij}$ , and  $b_{ik}^j$  are the enriched degrees of freedom associated with the shifted elastic asymptotic crack tip functions  $\tilde{F}_{ijk}(r, \theta)$ . As mentioned in Ref. [15], the existence of edge cracks is only effective on the distribution of  $u(x, y)$  and would not affect  $v(x)$ . That is why the special enrichment functions are just added to the axial displacement approximation in Eq. (4) [15]. The shifted enrichment functions in Eq. (4) are defined as follows:

$$\tilde{\phi}_{ij}(\xi) = \phi(\xi) - \phi(\xi_i^j) \quad (5)$$

$$\tilde{F}_{ijk}(r, \theta) = F_k(r, \theta) - F_k(r_i^j, \theta_i^j) \quad (6)$$

The application of these shifted functions is useful to eliminate the influence of the enrichment on the displacement of the nodes located on the crack tip and sides for ease of imposing boundary conditions [15].

In Eq. (5), the Heaviside function is defined by Eq. (7).

$$\phi(\xi) = \begin{cases} +1 & \text{if } \xi > 0 \\ -1 & \text{if } \xi < 0 \end{cases} \quad (7)$$

Also  $\phi(\xi_i^j)$  is the numerical value of this function in the  $i^j$ th degree of freedom. For modeling the structural discontinuity, due to the presence of crack, the shifted Heaviside function is added to the displacement field of the nodes located on the crack sides (degrees of freedom are represented by square in Fig. 1). Also  $F_k(r, \theta)$  in Eq. (6) is defined as Eq. (8) [15].

$$F_k(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \cdot \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \cdot \sin(\theta) \right\} \quad (8)$$

Also  $F_k(r_i^j, \theta_i^j)$  are the numerical values of this function in the  $i^j$ th degree of freedom. In the above equations,  $(r, \theta)$  are the polar coordinates located at the crack tip. In similar analogy to the application of  $\tilde{\phi}_{ij}(\xi)$ ,  $\tilde{F}_{ijk}(r, \theta)$  is added to the displacement field of the crack tip nodes (degrees of freedom are represented by triangle in Fig. 1).

### 3 STABILITY ANALYSIS OF STRUCTURES

The buckling problem of elastic structures is formulated by the following eigenvalue equation [16].

$$([K] + \lambda_j [K_g]) \{u_j\} = \{0\} \quad (9)$$

Where,  $[K]$  is the structure flexural stiffness matrix,  $[K_g]$  is the structure stability matrix or geometrical stiffness matrix,  $\lambda_j$  is the  $j^{\text{th}}$  eigenvalue of the system of Eq. (9) and  $\{u_j\}$  is related buckling mode shape in terms of the displacement of the computational nodes. As is well known, for a structure with  $n$  degrees of freedom there would be  $n$  solutions to the eigenvalue problem yielding  $n$  different  $\lambda$  and their associated displacement patterns  $\{u\}$ . However, in stability problems the lowest value of  $\lambda$  is of concern which corresponds to the lowest load factor that would cause the system to be unstable. To solve the eigenvalue problem of Eq. (9),  $[K]$  and  $[K_g]$  should be measured for the mentioned element. By using standard finite element procedure these structural matrixes can be found as follows [15].

$$[K] = \int \int \int [B]^T \cdot [D] \cdot [B] dV \tag{10}$$

Where  $[B]$  and  $[D]$  are defined as Eq. (11) and Eq. (12) [15]. In these equations,  $E$  is the elastic modulus,  $G$  is the shear modulus, and  $k$  is the shear correction factor.

The element geometric stiffness matrix is also introduced by Eq. (13) [15]. In Eq. (13),  $\sigma_{x_0}$  and  $\tau_{xy_0}$  are the axial and the shear pre-stresses in an elastic body. In addition the other parameters can be found as follows [15].

$$[B] = \begin{bmatrix} \frac{\partial N_i(x)}{\partial x} \cdot H_j(y) & 0 & \frac{\partial N_i(x)}{\partial x} \cdot \tilde{\phi}_{ij}(\xi) \cdot H_j(y) & \frac{\partial N_i(x)}{\partial x} \cdot \tilde{F}_{ijk}(r, \theta) \cdot H_j(y) \\ \frac{\partial H_j(y)}{\partial y} \cdot N_i(x) & \frac{\partial N_i(x)}{\partial x} & \frac{\partial H_j(y)}{\partial y} \cdot \tilde{\phi}_{ij}(\xi) \cdot N_i(x) & \frac{\partial H_j(y)}{\partial y} \cdot \tilde{F}_{ijk}(r, \theta) \cdot N_i(x) \end{bmatrix} \tag{11}$$

$i = 1 \dots 2, j = 1 \dots n, k = 1 \dots 4$

$$[D] = \begin{bmatrix} E & 0 \\ 0 & k \cdot G \end{bmatrix} \tag{12}$$

$$[K_g] = \sigma_{x_0} \cdot \int \int \int_0^L ([B_{NL}^1]^T \cdot [B_{NL}^1] + [B_{NL}^2]^T \cdot [B_{NL}^2]) dx dy dz + 2 \cdot \tau_{xy_0} \cdot \int \int \int_0^L ([B_{NL}^1]^T \cdot [B_{NL}^3]) dx dy dz \tag{13}$$

$$[B_{NL}^1] = \begin{bmatrix} \frac{\partial N_i(x)}{\partial x} \cdot H_j(y) & 0 & \frac{\partial N_i(x)}{\partial x} \cdot \tilde{\phi}_{ij}(\xi) \cdot H_j(y) & \frac{\partial N_i(x)}{\partial x} \cdot \tilde{F}_{ijk}(r, \theta) \cdot H_j(y) \end{bmatrix} \tag{14}$$

$$[B_{NL}^2] = \begin{bmatrix} [0]_{1 \times n} & \frac{\partial N_i(x)}{\partial x} & [0]_{1 \times n} & [0]_{1 \times (4n)} \end{bmatrix} \tag{15}$$

$$[B_{NL}^3] = \begin{bmatrix} \frac{\partial H_j(y)}{\partial y} \cdot N_i(x) & 0 & \frac{\partial H_j(y)}{\partial y} \cdot \tilde{\phi}_{ij}(\xi) \cdot N_i(x) & \frac{\partial H_j(y)}{\partial y} \cdot \tilde{F}_{ijk}(r, \theta) \cdot N_i(x) \end{bmatrix} \tag{16}$$

#### 4 NUMERICAL RESULTS AND DISCUSSION

Three examples are presented to illustrate the effectiveness and the efficiency of the proposed method. The general applicability of the method is evident from the variety of the offered examples.

##### 4.1 A Prismatic column with a single edge crack

This example is taken from paper published by Yazdchi and Gowhari [8]. Here, it is treated by the proposed method and the two results are compared. A column with rectangular cross section with a following data is considered.  $b = h = 0.1 \text{ m}$ ,  $L = 1.3 \text{ m}$ ,  $E = 200 \text{ GPa}$  and  $\nu = 0.3$ . The effect of crack depth and crack location for 3 different boundary conditions of the column is investigated. To compare the computational efficiency of the presented method with available standard finite elements models, a sensitivity analysis is

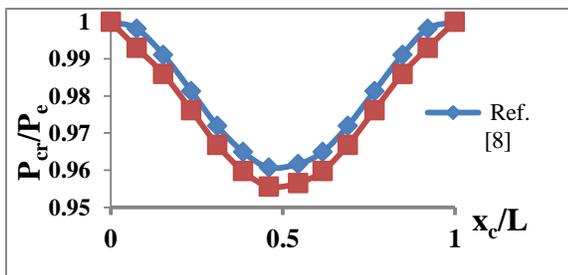
carried out to find appropriate mesh density for modeling the problem. A center cracked column with Fixed-Free boundary condition is considered for this analysis. Also the crack length is assumed to be  $a = 0.02 \text{ m}$ . The results can be found in table 1.

It should be noted that the results are presented in non-dimensional form by dividing the measured buckling load of cracked column by the Euler buckling load of the intact column. As it is evident from the table 1, the results are converged with much less number of DOFs in comparison with classical FEM results. It shows that the presented Extended Finite Element is a very powerful tool to model problems with a few computational efforts. Figures 2 to 10 show the results of proposed method and Ref. [8]. In these figures  $P_{cr}$  is the buckling load of cracked column. To show the results in non-dimensional form, this load is divided to the buckling load of intact column ( $P_e$ ). Also  $X_c$  is the

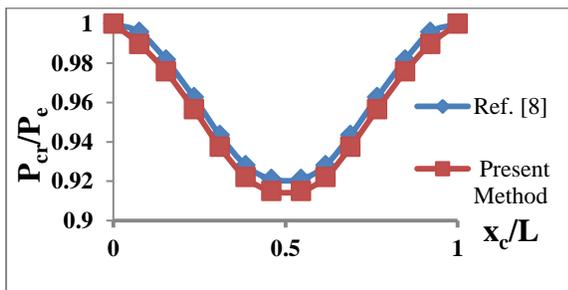
position of edge crack which is divided to column length. In addition  $a$  is the crack length and  $h$  is the width of column. As expected the deeper cracks causes the larger decreasing in the buckling capacity. The crack location has different effects depending on the column boundary conditions.

**Table 1** Convergence of measured buckling load for cracked column

FEM	DOFs	130	264	488	1024	2160
	$P_{cr}/P_e$	0.81	0.85	0.89	0.91	0.92
XFEM	DOFs	40	74	98	120	-
	$P_{cr}/P_e$	0.86	0.9	0.91	0.92	-



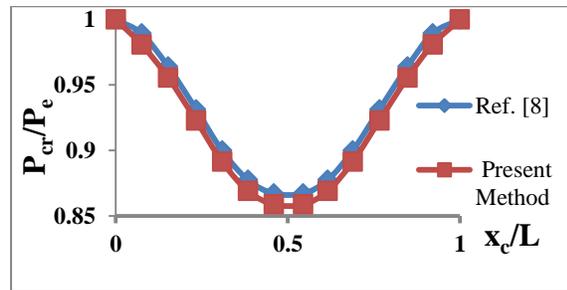
**Fig. 2** Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.2$  and Pinned- Pinned column



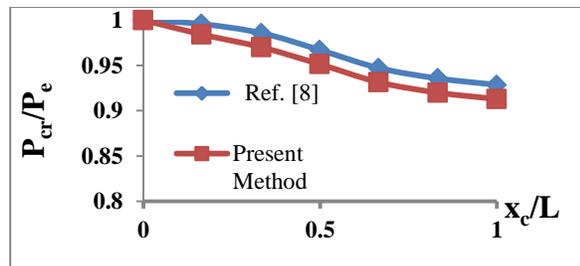
**Fig. 3** Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.3$  and Pinned-Pinned column

It is well-known that, the strain energy stored in an elastic body under bending is directly related to the magnitude of the bending moment. Therefore, for a constant crack depth, a crack located in the section of maximum bending moment causes the largest decrease in the critical buckling load. In addition a crack located in the inflexion points (moment zero points) has no effect on the critical buckling load. Comparison shows good agreement between the results of two works, although the difference between the results is more significant for the cracks with larger dimensionless

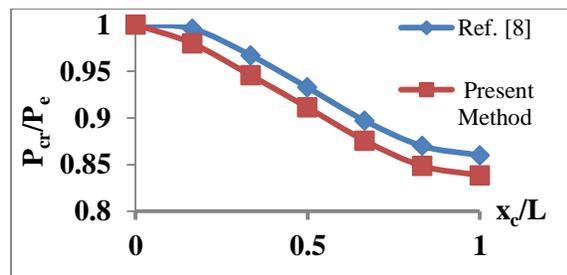
depth. The reason of this phenomenon will be discussed later.



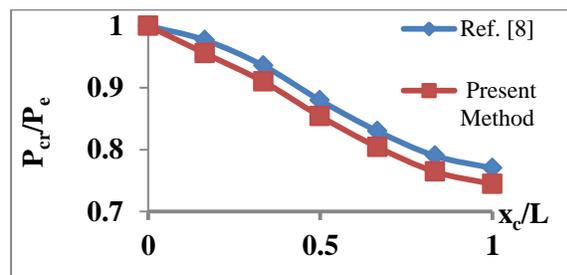
**Fig. 4** Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.4$  and Pinned- Pinned column



**Fig. 5** Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.2$  and Fixed- Free column



**Fig. 6** Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.3$  and Fixed- Free column



**Fig. 7** Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.4$  and Fixed- Free column

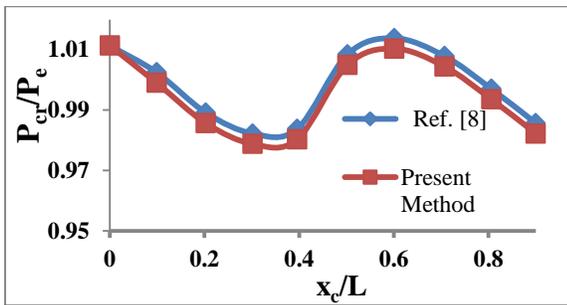


Fig. 8 Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.2$  and Fixed- Pinned column

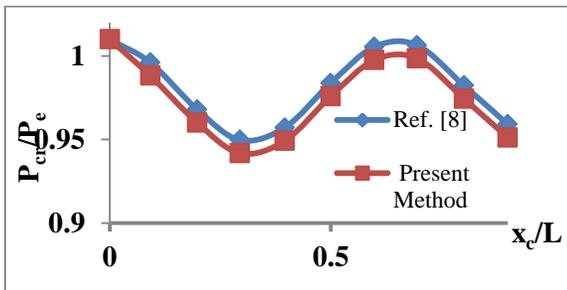


Fig. 9 Variation of  $P_{cr}/P_e$  versus the dimensionless crack location for  $a/h = 0.3$  and Fixed- Pinned column

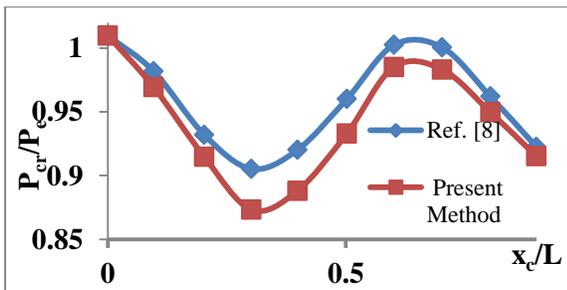


Fig. 10 Variation of  $P_{cr}/P_e$  versus, the dimensionless crack location for  $a/h = 0.4$  and Fixed- Pinned column

**4.2 A column with a single and double edge crack**

In this example, a rectangular column with a single and double edge crack having pinned-pinned and fixed-free boundary conditions with two different dimensionless crack depth ( $a/h = 0.15$  and  $a/h = 0.45$ ) is investigated. A column is considered with the following data.

$b = h = 0.1\text{ m}$ ,  $L = 1\text{ m}$ ,  $E = 200\text{ GPa}$  and  $\nu = 0.3$

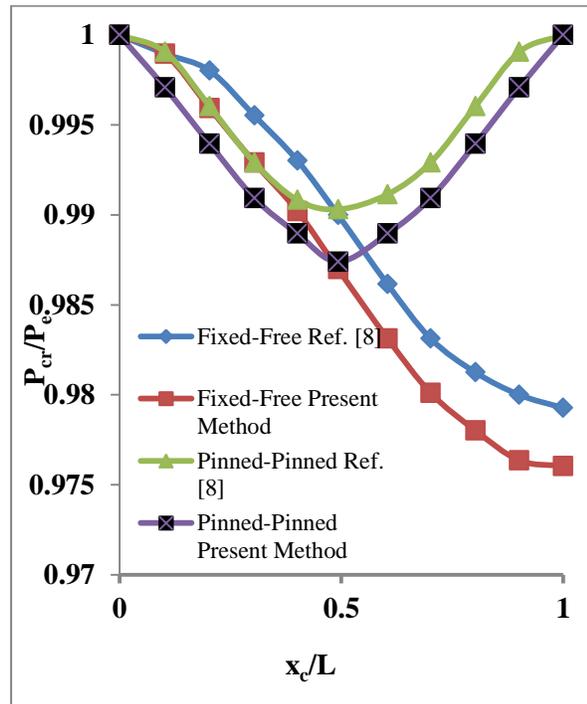


Fig. 11 Variation of  $P_{cr}/P_e$  versus, the crack location for a column having a single edge crack with  $a/h = 0.15$

Figs. 11 to 14 show the obtained results. The proposed results are compared with the results of Ref. [8]. The difference between two works is less than 1 percent for a column with the dimensionless crack depth of 0.15 for both single and double edge cracked column (Figs. 11 and 12). When  $a/h$  is increased to 0.45, the results are more different for a single cracked column, though the biggest difference is still less than 8 percent (Fig. 13). But for the last situation, the double edge cracked column with  $a/h$  equal to 0.45, two results are completely different (Fig. 14). According to the Ref. [8] the largest decrease in the buckling load is 40 percent for a fixed-free column and it is 22 percent for a pinned-pinned one.

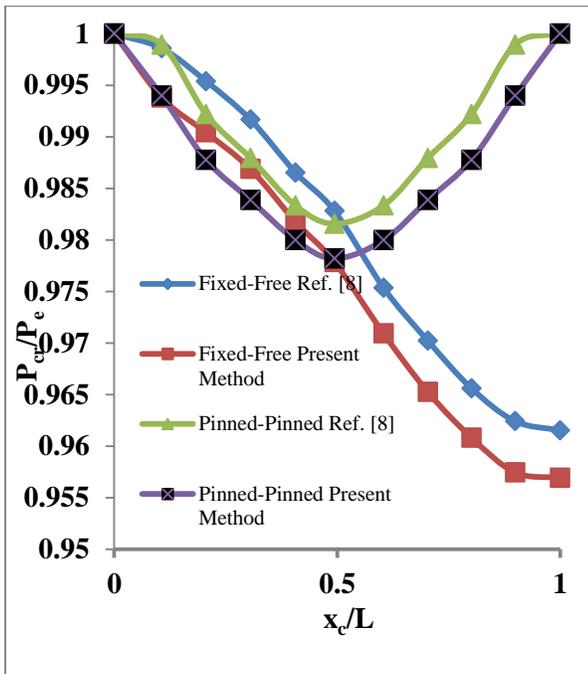


Fig. 12 Variation of  $P_{cr}/P_e$  versus, the crack location for a column having a double edge crack with  $a/h = 0.15$

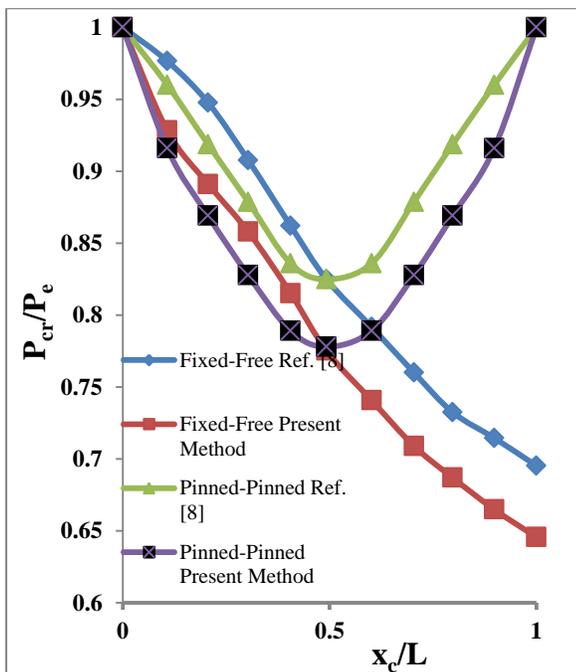


Fig. 13 Variation of  $P_{cr}/P_e$  versus, the crack location for a column having a single edge crack with  $a/h = 0.45$

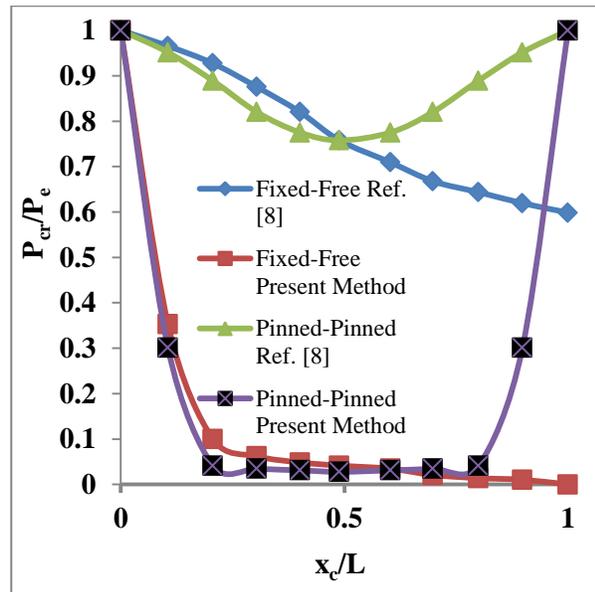


Fig. 14 Variation of  $P_{cr}/P_e$  versus the crack location for a column having a double edge crack with  $a/h = 0.45$

But the proposed results show that, the column lost almost all of its buckling capacity. Two previous examples show that, the two methods end in same results for short cracks with a  $a/h$  less than 0.3. But they make different results for deeper cracks especially for double edge cracked columns. It seems one or both methods cannot model the effect of deep cracks on the stability of columns. For more accurate judgment, two approaches must be evaluated by experimental results. It is done in the next example.

### 4.3. A fixed-pinned edge cracked column

Vanderveldt and Liebowitz [2] published some useful experimental data for the buckling of single and double notched fixed-pinned column. The crack was located in the middle of the column. These data are used here to compare with the rotational spring models and proposed numerical results. Fig. 15 shows the results for single edge cracked column. Two different analytical works those are published by Gruel and Kisa [6] and Yazdchi and Gowhari [8] have good agreement with proposed and experimental data when  $a/h$  is less than 0.3. But for deeper cracks the results are totally different. However, proposed results are close to the experiment for a wide range of crack dimensionless depth.

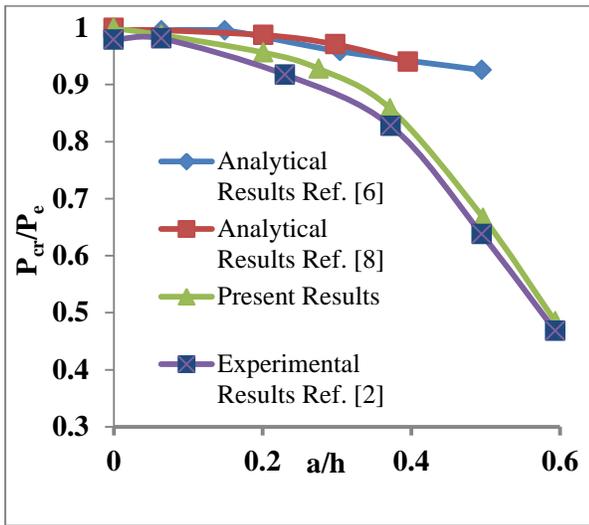


Fig. 15 Variation of  $P_{cr}/P_e$  versus, the dimensionless crack depth for a pinned-fixed column having a single edge crack

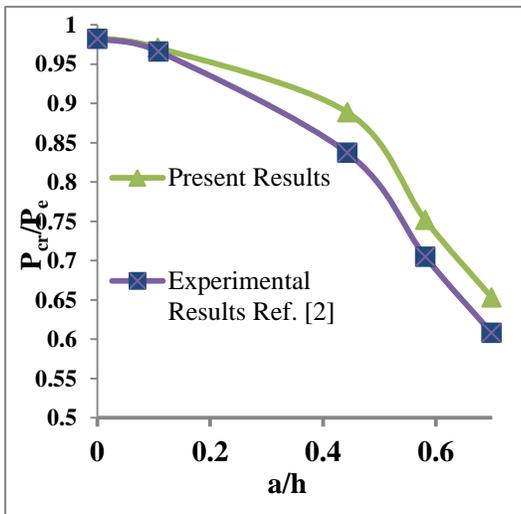


Fig. 16 Variation of  $P_{cr}/P_e$  versus, the dimensionless crack depth for a pinned-fixed column having a double edge crack

Fig. 16 shows the experimental and proposed results for double edge cracked column. Again the largest difference between two works is less than 8 percent. Unfortunately there is no analytical solution available for this case for comparing with the experimental and proposed results. However the difference between analytical and proposed methods for  $a/h$  more than 0.3 in previous examples, concludes that, the analytical methods based on modeling cracks as a rotational spring which has been widely used in the literature, cannot model deep cracks with  $a/h$  larger than 0.3. Maybe it is

because of the fact that, a suitable flexibility function for calculating the true reduced stiffness of a column is not available for large cracks yet.

## 5 CONCLUSIONS

The buckling analysis of slender prismatic cracked columns with rectangular section, subjected to concentrated vertical loads has been presented in this work using Extended Finite Element Method.

The most important conclusions can be noted as follows:

1. The effect of a crack on the buckling capacity of a column depends on the crack depth and its location.
2. In columns under concentrated axial compression, the existence of cracks decreases the buckling load. As expected, the load carrying capacity decreases as the crack depth increase. Moreover, the effect of crack location for different boundary conditions of the column is different. The largest decrease of the buckling load occurs when the crack is located at the position with maximum curvature of the buckling mode shape of the column. If a crack is located just in the inflexion points of the column, its effect on the buckling load is negligible.
3. The proposed results are in a very good agreement with analytical results based on rotational spring models available in the literature for  $a/h$  less than 0.3.
4. The available analytical methods based on modeling a crack as a massless rotational spring, end in a wrong results for cracks with dimensionless depth more than 0.3.
5. The proposed results are in a good agreement with the experimental data for a wide range of dimensionless crack depth.
6. The presented Extended Finite Element Model in Ref. [15] is an efficient and powerful method for analyzing the stability of cracked slender members.

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